Chap 9: Foundations of 2D CFT

9.1. Axioms 2D EQFT

We will work in $\mathbb{R}^{2,0} \simeq \mathbb{C}$
(Wick rotated $\mathbb{R}^{1,1}$)

Def: $M \overset{\text{open}}{\subseteq} \mathbb{C}$

- $\Phi : \mathcal{S}(M) \to S(\mathcal{H})$

(quantum fields)

- $G_{i_1 \dots i_n}(z_1, \dots, z_n) \equiv \langle \Omega | \underbrace{\left(\prod_{j=1}^{n} \Phi_{i_j}(z_j)\right)}_{\text{time ordered}} | \Omega \rangle$

(n-point function / correlation)

$G_{i_1 \dots i_n} : M_n \to \mathbb{C}$ s/t

anal cont from $\mathcal{S}(\mathbb{R}^{2n})$

s/t $M_n \equiv \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j \}$

← configuration space

Def: $\cdot\ M_n^- = \{(z_1,\dots,z_n)\in M_n \mid \mathrm{Re}\, z_j > 0\}$

$\cdot\ \mathcal{S}_n^+ = \{ f \in \mathcal{S}(\mathbb{C}^n) \mid \mathrm{Supp}(f) \subseteq M_n^+ \}$

$(\Rightarrow \mathcal{S}_0^+ \simeq \mathbb{C})$

$\cdot\ z = t + iy\ ,\ \bar{z} = t - iy\ \ (t, y \in \mathbb{R})$

indep

| The isometry group of 2DQFT is $\underset{\underset{U(1)}{\text{is}}}{SO(2)} \ltimes \mathbb{R}^2 \equiv E(2)$

(Euclidean group in 2D)

If furthermore we want conf sym. we add dilation and SCT.

OS-axioms for 2D (E)QFT:

Let $B_0 \in \mathrm{Set}^{\mathrm{count}}$

$B = \bigcup_{n=0}^{\infty} (B_0)^n$ then the QFT is described by $\left\{ G_{i_1 \dots i_n} : M_n \longrightarrow \mathbb{C} \right\}^{n \geq 0}_{(i_1 \dots i_n) \in B_0^n}$

A1 (Locality). $\forall \sigma \in S_n,\ (i_1 \dots i_n) \in B_0^n$
$(z_1, \dots, z_n) \in M_n$

$$G_{i_1 \dots i_n}(z_1, \dots, z_n) = G_{i_{\sigma(1)} \dots i_{\sigma(n)}}(z_{\sigma(1)}, \dots, z_{\sigma(n)})$$

A2 (Covariance): $\forall i_j \in B_0$

$\exists h_{i_j}, \bar{h}_{i_j} \in \mathbb{R}$ s/t $\forall w \in E(2), n \geq 1$

$$G_{i_1 \dots i_n}(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$$ eval after derivative.

$$= \prod_{j=1}^{n} \left(\frac{dw(z_j)}{dz} \right)^{h_{i_j}} \left(\overline{\frac{dw(z_j)}{dz}} \right)^{\bar{h}_{i_j}}$$

$$G_{i_1 \dots i_n}(w(z_1), \overline{w(z_1)}, \dots, w(z_n), \overline{w(z_n)})$$

$h_{i_j} \equiv h_j$ since s_j gives the exponent of rotation generator.

$s_i \equiv h_i - \bar{h}_i$: conf spin
$d_i \equiv h_i + \bar{h}_i$: scaling dim. / conf dim.

1. • We demand G depends on $\bar{z}$ because this comes from the fact: as complex manifolds, identifying $\mathbb{C} \simeq \mathbb{R}^{2,0}$ require $\mathbb{R}^{2,0}$ has a complex structure. This is only given by complexifying $\mathbb{R}^{2,0} \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathbb{R}^2 \oplus \overline{\mathbb{R}^2} = \mathbb{C}^2$ which gives chiral and antichiral part by complex structure $J^2 = -\mathrm{id}_{\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}}$

• We usually omit the $\bar{z}$ part to simplify

(So it's only chiral instead of full CFT

- Given covariance we can classify 2,3-pt function:

By using translation cov, redefine $z_{ij} = z_i - z_j$, then we can set

$$G_{i_1 i_2}(z_1, \bar{z}_1, z_2, \bar{z}_2) = G_{i_1 i_2}(z_{12}, \bar{z}_{12}, 0, 0)$$

$$G_{i_1 i_2}(z_1, \bar{z}_1, z_2, \bar{z}_2) = C\, z_{12}^{-h_1 - h_2}\, \bar{z}_{12}^{-\bar{h}_1 - \bar{h}_2}$$

(for arbitrary $h_i, \bar{h}_i$) (2)

$$G_{i_1 i_1}(z_1, \bar{z}_1, z_2, \bar{z}_2) = -\log |z_{12}|^2 \quad (2')$$

(for $h = \bar{h}$)

Check that for $w(z) = e^{i\theta} z$

$$C_{i_1 i_2} z_{12}^{-h_1 - h_2} \bar{z}_{12}^{-\bar{h}_1 - \bar{h}_2} = (e^{i\theta})^{h_1 + h_2} (e^{-i\theta})^{\bar{h}_1 + \bar{h}_2} C$$

$$= C (e^{i\theta})^{-h_1 - h_2} (e^{-i\theta})^{-\bar{h}_1 - \bar{h}_2} z_{12}^{-h_1 - h_2} \bar{z}_{12}^{-\bar{h}_1 - \bar{h}_2}$$

and

$$-\log|z_{12}|^2 = (e^{i\theta})^{2h}(e^{-i\theta})^{2h} - \log|z_{12}|^2$$

- A general form of 3 point function is

$$G_{i_1 i_2 i_3}(z_1, \bar{z}_1, \dots, z_3, \bar{z}_3) \qquad (3)$$

$$= C_{i_1 i_2 i_3} \sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{3} z_{ij}^{-h_i - h_j + h_k} + \sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{3} \bar{z}_{ij}^{-\bar{h}_i - \bar{h}_j + \bar{h}_k}$$

$$G_{\underbrace{i_1 \dots i_1}_{2n \text{ times}}} = \frac{k^n}{2^n n!} \sum_{\sigma \in S_{2n}} \prod_{j=1}^{n} (z_{\sigma(j)} - z_{\sigma(n+j)})^{-2h_1} \qquad (n)$$

where $\bar{h}_1 = 0$

Let $\underline{g}^+ = \bigoplus_{n=0}^{\infty} g_n^+$ mit $\underline{f} = (f_i)_{i\in B} \in \underline{g}^+$

for $0 \neq f_i \in \mathcal{S}_n^+$ for at most finitely many, $i \in B_0^n$

A 3 (Reflection positivity)

$\exists * : B_0 \to B_0$, $*^2 = id_{B_0}$

$\Rightarrow * : B \to B$ s/t
$i \mapsto i^*$

1, $G_i(z) = G_{i^*}(\Theta(z)) = G_{i^*}(-z^*)$

s/t z^*: complex conj of z

2, $\langle \underline{f}, \underline{f} \rangle \geq 0 \quad \forall \underline{f} \in \underline{\mathcal{S}}^+$ for

$$\langle \underline{f}, \underline{f} \rangle = \sum_{i,j \in B} \sum_{n,m \geq 0} \int_{M_{n+m}} G_{i^* j}(\Theta(z_1), \ldots, \Theta(z_n), w_1, \ldots, w_n) f_i(z)^* f_j(w) \, d^n z \, d^n w$$

Lem: OS3 gives $\langle -, - \rangle$ and that's a pos-semi def bilinear form on $\underline{\mathcal{S}}^+$

$\Rightarrow \mathbb{H} = \underline{g}^{+} / \operatorname{Ker} \langle -,- \rangle$

So for $j \in B_0$

$\Phi_j(f)([\underline{g}]) = [\underline{g} \times f] = (g_i)_{i \in B_0^n} \times f$

$= (\underline{g} \times f)_{i_1 \ldots i_{n+1}}$ for

$(\underline{g} \times f)_{i_1 \ldots i_{n+1}}(z_1, \ldots, z_{n+1})$

$= g_{i_1 \ldots i_n}(z_1, \ldots, z_n) f(z_{n+1}) \delta_{j, i_{n+1}}$

We can show this satisfies field conditions with vacuum

$\Omega = (1, 0, 0 \ldots)$ and thus W axioms.

! A3 implies that S matrix is unitary and thus the QFT is unitary

9.2. CFT and $T^{\mu\nu}$

Def: If (G_i, B, H): 2DEQFT then it's a 2D CFT if satisfies these extra axioms

Axiom 4: (Scaling covariance)

$$G_i(z_1,\dots,z_n) = (e^{\tau})^{\sum_{j=1}^{n} h_j + \bar{h}_j} G_i(e^{\tau}z_1,\dots,e^{\tau}z_n)$$

for $i \in B_0^n$, $\tau \in \mathbb{R}$, $(z_1,\dots,z_n) \in M$.

Axiom 2+4 gives Conf / Möbius covariance

! We leave out SCT is because

in 2DQFT, dilation / scale inv
$\Rightarrow$ SCT inv by Zamolodchikov-thm
up to some technical assumptions
(which we will assume to be true for our case)
unitary, discrete spec in scale dim, $\exists$ scale current, unbroken scale sym.

(String worldsheet theory violate unitary, discrete spec)

Lem: In 2D CFT any 2-point func has the form of (2)

$$G_{ij} = C_{ij}\, z_{12}^{-h_i + h_j}\, \bar{z}_{12}^{-\bar{h}_i + \bar{h}_j} \quad (z_{12} = z_1 - z_2)$$

for $C_{ij} \in \mathbb{C}$

Any 3-point in CFT is of the form (3)

So 2 and 3-pt correlators are completely determined by C_{ij}, C_{ijk}

prf: Given arbitrary transform $z \mapsto e^{\tau + i\alpha} z$

$$G_{ij}(z_1, z_2) \equiv G_{ij}(z_{12}, 0) \equiv G(z, 0)$$

$$= (e^{\tau + i\alpha})^{h_i + h_j} (e^{\tau - i\alpha})^{\bar{h}_j + \bar{h}_i} G(e^{\tau + i\alpha} z, 0)$$

Let $z = e^{\tau + i\alpha}$

$$\Rightarrow G(1,0) = z^{h_i + h_j}\, \bar{z}^{\bar{h}_i + \bar{h}_j}\, G(z, 0)$$

$$\Rightarrow C_{ij} = G(1,0)$$

Similar for 3pt-funcs.

Prop ("Conformal Ward identities")

$G \in CFT$ then

$$0 = \sum_{j=1}^{n} \left(z_j^{m+1} \partial_{z_j} + (m+1) h_j z_j^{m} \right) G(z_1, \dots, z_n)$$

for $m \in \{0, 1, 2\}$.

prf: Use covariance wrt

$w(z) = z / (1 - \alpha z^{m+1})$

then apply $\left.\frac{d}{d\alpha}\right|_{\alpha=0}$

! From CWI

$G(z_1, z_2, z_3, z_4)$

$$= F(r(z), \overline{r(z)}) \prod_{i<j} z_{ij}^{-(h_i+h_j)+\frac{1}{3}h} \prod_{i<j} \bar{z}_{ij}^{-(\bar{h}_i+\bar{h}_j)+\frac{1}{3}\bar{h}} \quad (4)$$

for $r(z) \equiv (r_{12} r_{34}) / (z_{13} z_{24})$, F: holo

n-point funcs are monomials in $z_{ij}, \bar{z}_{ij}$ and functions like F.

Axiom 5 ($\exists\, T^{\mu\nu}$)

Given CFT_2, $\exists\, T_{\mu\nu}$: tensorfield
That is, there are 4 scalarfields s/t.

1, $T_{\mu\nu} = T_{\nu\mu}$, $T_{\mu\nu}(z)^* = T_{\nu\mu}(\Theta(z))$

2, $\partial_0 T_{\mu 0} + \partial_1 T_{\mu 1} = 0$ $\quad (\partial_0 = \partial_t, \partial_1 = \partial_y)$

3, $d(T_{\mu\nu}) \equiv h_{\mu\nu} + \overline{h_{\mu\nu}} = 2 \quad \forall \mu, \nu \in \{0,1\}$
conf dim

$s(T_{00} - T_{11} \pm 2i T_{01}) = \pm 2$
conf spin

Thm (Lüscher-Mack): Axiom 1-5 imply
1, $\mathrm{tr}(T_{\mu\nu}) = T^{\mu}{}_{\mu} = 0$

$\Rightarrow T \equiv T_{00} - i T_{01} = \frac{1}{2}(T_{00} - T_{11} - 2i T_{01})$ is indep of $\bar{z}$ ($\partial_{\bar{z}} T = 0$) and

$\bar{T} \equiv T_{00} + i T_{01}$ has $\partial_z \bar{T} = 0$

$\Rightarrow$ T: holomorphic, $\bar{T}$: antiholo.

$h(T) = \bar{h}(\bar{T}) = 2$, $\bar{h}(T) = h(\bar{T}) = 0$

2. for $n \in \mathbb{Z}$

$$L_n \equiv \frac{1}{2\pi i} \oint_{|z|=1} T(z)\, z^{-n-1}\, dz$$

$$\bar{L}_n = \frac{1}{2\pi i} \oint_{|z|=1} \bar{T}(z)\, z^{-n-1}\, dz$$

then $\{L_n\}_{n\in\mathbb{N}}, \{\bar{L}_n\} \in \mathcal{U}(D \overset{\text{dense}}{\subseteq} H)$

and define $\mathbb{H} \in$ Vir-rep, $\overline{\text{Vir}}$-rep

1. We can define $T, \bar{T}$ in terms of formal power series

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \, \bar{z}^{-n-2}$$

- $\{L_n\}, \{\bar{L}_n\}$ gives conf weight.

$$\bigoplus \mathrm{Vir}(c,h) \otimes \mathrm{Vir}(c,\bar{h}) \in \mathrm{Vir}\times\overline{\mathrm{Vir}}\text{-rep}$$

This rep is called minimal if $\bigoplus$: finite